MULTIPLIER METHODS FOR SADDLE POINTS

by

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A Thesis

submitted to

the Graduate School of

The Chinese University of Hong Kong

(Division of Mathematics)

In Partial Fulfillment

of the Requirements for the Degree of

Master of Philosophy (M. Phil.)

HONG KONG

May 1978

THE CHINESE UNIVERSITY OF HONG KONG

Graduate School

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ACKNOWLEDGEMENT

I would like to thank Dr. Wai-leung Chan for his supervision. Thanks are also due to Mr. Billy Lam for his typing.

SECTION 1. INTRODUCTION

The aim of this thesis is to give an anlysis to the saddle point problem. If f is a function from $E^n \times E^m$ into E', a point $(\bar{x}, \bar{y})$ is called a saddle point for f, if for $x \in E^n$, $y \in E^m$, the inequality

$$f(\bar{x}, y) \leq f(\bar{x}, \bar{y}) \leq f(x, \bar{y})$$

holds. The saddle point problem is the determination of a saddle point of a given function provided that such a point exists. We first give the necessary and sufficient condition for the problem without constraint and with constraints. Then we derive a multiplier method for searching saddle points of f with equality constraints. The investigation is in the spirits of the augmentability due to Hestnes [6], and the development of the multiplier method is an extension of the work of D.P. Bertsekas [2], who considered the minimum case.

Contrary to the study of optimization problem, the study of saddle point problems have taken up by only relatively few research works, which include V.F. Demyanov, V.N. Malozemov [3], [4], [5] and A. Auslender [1], who views the saddle point problem as a special case of a mini-max problem. Numerical methods for solving the problem are studied by adapting methods employed in non-linear programming, such as the gradient method, gradient projection method, penalty method, etc., [3], [1], [7]. It has been proved that these methods do converge. However, results on rate of convergence are seldom discussed. This thesis started from the viewpoint of augmentability and proved that there is a multiplier method for the saddle point problem. The convergence of the method is proved and the rate of convergence is given.

SECTION 2.

If $(\bar{x}, \bar{y})$ is a saddle point for f, then, we have

$$f(\bar{x}, \bar{y}) \leq f(x, \bar{y}) \qquad \forall\, x \in E^n$$

i.e. $\bar{x}$ minimize $f(\cdot\,, \bar{y})$ on E^n. Similarly,

$$f(\bar{x}, y) \leq f(\bar{x}, \bar{y}) \qquad \forall\, y \in E^m .$$

Therefore, $\bar{y}$ maximize $f(\bar{x}, \cdot\,)$ on E^m. From this point of view, we see immediately that a saddle point problem can be dealt with by separating it into a minimizing problem and a maximizing one. The usual method of analysis of optimization problem can now be applied to our problem.

<u>Proposition 1</u>. Suppose f is C^1, i.e. the partial derivatives of f with respect to the variables $x^1, \ldots, x^n; y^1, \ldots, y^m$ exist and are continuous. In order that $(\bar{x}, \bar{y})$ be a saddle point of f, it is necessarily that

$$\nabla_x f(\bar{x}, \bar{y}) = 0 , \qquad \nabla_y f(\bar{x}, \bar{y}) = 0 . \tag{1}$$

The proof is immediate. Condition (1) is not sufficient, for we see that the function $f(x, y) = x^3 - y^3$, $(x, y) \in E^2$ has the property $\nabla_x f(0, 0) = 0$, $\nabla_y f(0, 0) = 0$. But $(0, 0)$ is not a saddle point for f. To obtain sufficient conditions, we should go on to the second order terms. Consider the Taylor's expansion of $f(\cdot\,, \bar{y})$:

$$f(x, \bar{y}) - f(\bar{x}, \bar{y}) = \nabla_x f(\bar{x}, \bar{y})(x - \bar{x}) + \tfrac{1}{2}(x - \bar{x})^T \nabla^2_{xx} f(\bar{x} + \theta(x - \bar{x}), \bar{y})(x - \bar{x}),$$

where $0 \leq \theta \leq 1$ and x^T denote the transpose of x. Since $\nabla_x f(\bar{x}, \bar{y}) = 0$, and $f(x, \bar{y}) \geq f(\bar{x}, \bar{y})$, we have

$$\tfrac{1}{2}(x - \bar{x})^T \nabla^2_{xx} f(\bar{x} + \theta(x - \bar{x}), \bar{y})(x - \bar{x}) \geq 0 . \tag{2}$$

We note that, if the matrix $\nabla^2_{xx} f(\bar{x}, \bar{y})$ is positive definite, i.e. for each $h \in E^n$, $h \neq 0$, we have $h^T \nabla^2_{xx} f(\bar{x}, \bar{y})h > 0$. Then by continuity, for x sufficiently near $\bar{x}$, inequality (2) holds. Consequently, $\bar{x}$ is a local minimum of $f(\cdot, \bar{y})$ if $\nabla_x f(\bar{x}, \bar{y}) = 0$ and $\nabla^2_{xx} f(\bar{x}, \bar{y})$ is positive definite. Similarly, if $\nabla_y f(\bar{x}, \bar{y}) = 0$ and $\nabla^2_{yy} f(\bar{x}, \bar{y})$ is negative definite, then $\bar{y}$ is a local maximum for $f(\bar{x}, \cdot)$. Thus we have

<u>Proposition 2</u>. Suppose f is C^2. If the following conditions hold, then $(\bar{x}, \bar{y})$ is a local saddle point for f:

(i) $\nabla_x f(\bar{x}, \bar{y}) = 0$, $\nabla_y f(\bar{x}, \bar{y}) = 0$;

(iia) $\nabla^2_{xx} f(\bar{x}, \bar{y})$ is positive definite;

(iib) $\nabla^2_{yy} f(\bar{x}, \bar{y})$ is negative definite.

<u>Remark</u>: Concerning condition (ii), if $\nabla^2_{xx} f(\bar{x}, \bar{y})$ is positive semi-definite within some open ball $B(\bar{x}\,;\varepsilon)$ centered at $\bar{x}$, then (2) is valid. Similarly, if $\nabla^2_{yy} f(\bar{x}, \bar{y})$ is negative semidefinite within some open ball $B(\bar{y}\,;\varepsilon')$ centered at $\bar{y}$, $\bar{y}$ is a local maximum for $f(\bar{x}, \cdot)$. In other words, proposition 2 is valid if condition (ii) is replaced by the following condition (ii)':

If there exists a positive number ε , such that

(a) $\nabla^2_{xx} f(x, \bar{y})$ is positive semidefinite for all $x \in B(\bar{x} ; \varepsilon)$;

(b) $\nabla^2_{yy} f(\bar{x}, y)$ is negative semi-definite for all $y \in B(\bar{y} ; \varepsilon)$.

To many problems of interest, the region in which we are asked to find the saddle point is governed by constraints. Here we are concerned mainly with equality constraints. More precisely, we want to locate a saddle point of f subject to the constraints

$$
\begin{aligned}
g_i(x) &= 0 \qquad\qquad i = 1, \ldots, k \ , \\
h_j(y) &= 0 \qquad\qquad j = 1, \ldots, \ell \ ,
\end{aligned}
\tag{P}
$$

where $g_i : E^n \to E^1$, $h_j : E^m \to E^1$ for all i, j . We see that if $(\bar{x}, \bar{y})$ is a saddle point subject to $g_i(x) = 0, h_j(y) = 0$, then $\bar{x}$ is a minimum point for $f(\cdot , \bar{y})$ subject to $g_i(x) = 0$ and $\bar{y}$ is a maximum point for $f(\bar{x}, \cdot)$ subject to $h_j(y) = 0$. Conversely, if $\bar{x}$ is a minimum of $f(\cdot , \bar{y})$ subject to $g_i(x) = 0$ and $\bar{y}$ is a maximum of $f(\bar{x}, \cdot)$ subject to $h_j(y) = 0$, then $(\bar{x}, \bar{y})$ is a saddle point of f subject to $g_i(x) = 0$ and $h_j(y) = 0$. According to the following well known

<u>Lemma 1</u>: Suppose $f, g_i : E_n \to E^1$ and C^1 , $\bar{x}$ minimizes f subject to $g_i(x) = 0$, and if $\{ \nabla g_1(\bar{x}), \ldots, \nabla g_k(\bar{x}) \}$ is linearly independent, then there exists multipliers $\lambda_1, \ldots, \lambda_k$, not all zero, s.t.

$$\nabla f(\bar{x}) + \lambda_1 \nabla g(\bar{x}) + \ldots + \lambda_k \nabla g_k(\bar{x}) = 0 \ ,$$

we have the following proposition:

<u>Proposition 4</u>. Suppose $f : E^n \times E^m \to E'$, $g_i : E^n \to E^1$, $h_j : E^m \to E^1$ are C^1 ; $(\bar{x}, \bar{y})$ is a saddle point for f subject to constraints $g_i(x) = 0$ and $h_j(y) = 0$, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant \ell$. The sets $\{\nabla g_1(\bar{x}), \ldots, \nabla g_k(\bar{x})\}$, $\{\nabla h_1(\bar{y}), \ldots, \nabla h_\ell(\bar{y})\}$ are linearly independent, then there exist two sets of multipliers $\bar{\lambda}_1, \ldots, \bar{\lambda}_k$; $\bar{\mu}_1, \ldots, \bar{\mu}_\ell$ both of them not all zero, s.t.

$$\begin{aligned} &\nabla_x f(\bar{x}, \bar{y}) + \bar{\lambda}_1 \nabla g_1(\bar{x}) + \ldots + \bar{\lambda}_k \nabla g_k(\bar{x}) = 0 , \\ &\nabla_y f(\bar{x}, \bar{y}) - \bar{\mu}_1 \nabla h_1(\bar{y}) - \ldots - \bar{\mu}_\ell \nabla h_\ell(\bar{y}) = 0 . \end{aligned} \tag{3}$$

If we let $F(x, y) = f(x, y) + \bar{\lambda}_1 g_1(x) + \ldots + \bar{\lambda}_k g_k(x) - \bar{\mu}_1 h_1(y) - \ldots - \bar{\mu}_\ell h_\ell(y)$ then equalities (3) are equivalent to $\nabla F(\bar{x}, \bar{y}) = 0$.

<u>Remark</u>: If $(\bar{x}, \bar{y})$ satisfies $g_i(\bar{x}) = 0$, $1 \leqslant i \leqslant k$, and $h_j(\bar{y}) = 0$, $1 \leqslant j \leqslant \ell$; and $(\bar{x}, \bar{y})$ is an unconstrained saddle point for F, then we can solve our problem by solving the system of equations:

$$\left\{ \begin{aligned} &\nabla_x f(x, y) + \sum_{i=1}^{k} \lambda_i \nabla g(x) = 0 , \\ &\nabla_y f(x, y) - \sum_{j=1}^{\ell} \mu_j \nabla h_j(y) = 0 , \\ &\qquad\qquad g_i(x) = 0 , \\ &\qquad\qquad h_j(y) = 0 . \end{aligned} \right. \tag{4}$$

In general, even in the optimization case which can be viewed as a special case of saddle point problem, a solution to the system (4) may not be a solution to our constrained problem. From proposition 2 and the remark following it, in order that $(\bar{x}, \bar{y})$ affords an unconstrained saddle point for F , we should have that the matrices $\nabla^2_{xx} F(\bar{x}, \bar{y})$ be positive definite and $\nabla^2_{yy} F(\bar{x}, \bar{y})$ be negative definite locally in addition to $\nabla F(\bar{x}, \bar{y}) = 0$. However, the function F so defined may not have a saddle point. It is due to the fact that F is not convex-concave enough to have a saddle point. We can made it so by adding to it a penalty function. By a penalty function, we mean $\frac{c}{2}[\sum_{i=1}^{k} g_i(x)^2 - \sum_{j=1}^{\ell} h_j(y)^2]$, with $c > 0$. We now define the new function:

$$L(x, y; c) = f(x, y) + \sum \bar{\lambda}_i g_i(x) - \sum \bar{\mu}_j h_j(y) + \frac{c}{2}[\sum (g_i(x)^2 - h_j(y)^2)] .$$

The sufficient condition for $(\bar{x}, \bar{y})$ to be a saddle point of $L(x, y; c)$ is

(i) $\nabla_x L(\bar{x}, \bar{y}; c) = 0$ and $\nabla_y L(\bar{x}, \bar{y}; c) = 0$,

(ii) The matrix $\nabla^2_{xx} L(\bar{x}, \bar{y}; c)$ is positive definite, and $\nabla^2_{yy} L(\bar{x}, \bar{y}; c)$ is negative definite.

Note that $\nabla_x L(\bar{x}, \bar{y}; c) = \nabla_x F(\bar{x}, \bar{y}) + c \sum g_i(\bar{x}) \nabla g_i(\bar{x})$. Since $g_i(\bar{x}) = 0$ for all i , hence $\nabla_x L(\bar{x}, \bar{y}; c) = 0$ is equivalent to $\nabla_x F(\bar{x}, \bar{y}) = 0$. Similarly, $\nabla_y L(\bar{x}, \bar{y}; c) = 0$ is the same as $\nabla_y F(\bar{x}, \bar{y}) = 0$.

For $\nabla^2_{xx} L(\bar{x}, \bar{y}; c)$, we have

$$\nabla^2_{xx} L(\bar{x}, \bar{y}; c) = \nabla^2_{xx} F(\bar{x}, \bar{y}) + c \sum_{i=1}^{k} \nabla g_i(\bar{x}) \nabla g_i(\bar{x})^T .$$

Since $\nabla^2_{xx} L(\bar{x}, \bar{y}; c)$ is required to be positive definite, i.e. for $h \in E^n$, $h \neq 0$, $h^T \nabla^2_{xx} L(\bar{x}, \bar{y}; c)h > 0$, thus we have, in particular, if $h^T \nabla g_i(\bar{x}) \nabla g_i(\bar{x})^T h = 0$, or equivalently, $\nabla g_i(\bar{x})^T h = 0$, then $h^T \nabla^2_{xx} F(\bar{x}, \bar{y})h > 0$. But, the converse may not hold, i.e. if $h^T \nabla_{xx^2} F(\bar{x}, \bar{y})h > 0$ for those $h \in E^n$, $h \neq 0$ and $\nabla g_i(\bar{x})^T h = 0$, $1 \leq i \leq k$, then it may happen that $\nabla^2_{xx} L(\bar{x}, \bar{y}; c)$ is not positive definite. Due to the following lemma ([6] p.261), it will be the case if c is sufficiently large.

<u>Lemma 5</u>. If C is a convex cone and $P(h)$, $Q(h)$ are two quadratic forms defined on C with the property that $Q(h) \geq 0$; $P(h) > 0$ whenever $Q(h) = 0$, then there is a constant $\bar{c} > 0$, such that for all $c \geq \bar{c}$, $P(h) + cQ(h) > 0$ on C.

If we let $Q(h) = h^T(\Sigma \nabla g_i(\bar{x}) \nabla g_i(\bar{x})^T)h$, $P(h) = h^T \nabla^2_{xx} F(\bar{x}, \bar{y})h$, then there is a $\bar{c} > 0$, s.t. for all $c \geq \bar{c}$,

$$h^T(\nabla^2_{xx} F(\bar{x}, \bar{y}) + c\, \Sigma \nabla g_i(\bar{x}) \nabla g_i(\bar{x})^T)h > 0 \quad .$$

Similarly, if for $k \in E^n$, $k \neq 0$, we have $k^T \nabla_{yy^2} F(\bar{x}, \bar{y})k < 0$ for all k s.t. $k^T(\Sigma \nabla h_j(\bar{y}) \nabla h_j(\bar{y})^T)k = 0$, then there is a $\bar{\bar{c}} > 0$, s.t. $\forall c \geq \bar{\bar{c}}$,

$$k^T(\nabla_{yy^2} F(\bar{x}, \bar{y}) - c\Sigma \nabla h_j(\bar{y}) \nabla h_j(\bar{y})^T)k < 0 \ .$$

In summary, we have proved the following

<u>Proposition 6</u>. If f, g_i, h_j are C^2, and that they satisfy the following conditions at $(\bar{x}, \bar{y})$, with $g_i(\bar{x}) = 0$, $h_j(\bar{y}) = 0$, for all i, j:

(i) $\nabla_x F(\bar{x}, \bar{y}) = 0$; $\nabla_y F(\bar{x}, \bar{y}) = 0$.

(ii) $h^T \nabla^2_{xx} F(\bar{x}, \bar{y})h > 0$ for $h \in E^n$, $h \neq 0$ and $\nabla g_i(\bar{x})^T h = 0$,

$k^T \nabla^2_{yy} F(\bar{x}, \bar{y})k < 0$ for $k \in E^m$, $k \neq 0$ and $\nabla h_j(\bar{y})^T k = 0$,

where $F(x, y) = f(x, y) + \Sigma \bar{\lambda}_i g_i(x) - \Sigma \bar{\mu}_j h_j(y)$. Then there is a constant $\bar{c} > 0$, s.t. for all $c \geq \bar{c}$, $(\bar{x}, \bar{y})$ is a local saddle point of

$$L(x, y; c) \equiv f(x, y) + \Sigma \bar{\lambda}_i g_i(x) - \Sigma \bar{\mu}_j h_j(y) + \frac{c}{2}[\Sigma g_i^2(x) - \Sigma h_j^2(y)] .$$

In particular, if $(x, y) \in E^n \times E^m$ satisfies the constraints $g_i(x) = 0$ and $h_j(y) = 0$, then $L(x, y; c) = f(x, y)$, and hence $(\bar{x}, \bar{y})$ affords a local saddle point for f subject to the constraints $g_i(x) = 0$ and $h_j(y) = 0$.

Based on this proposition, if $L(x, y; c)$ has a saddle point $(x(c), y(c))$ for each $c > 0$, then an algorithm for searching a constrained saddle point of f can be derived. Let $\{c_k\}$ be a sequence of positive numbers with the property that $c_k \to \infty$ as $k \to \infty$. If for some k, $(x(c_k), y(c_k))$ satisfies the constraints, then it is also a constrained saddle point for f ; if it is not the case, we proceed on $k + 1$, and check whether or not $(x(c_{k+1}), y(c_{k+1}))$ satisfies the constraints. If the sequence of saddle points $(x(c_k), y(c_k))$ has a limit point (x^*, y^*), then (x^*, y^*) is the constained saddle point for f as the following proposition shows:

<u>Proposition 7</u>. If (x^*, y^*) is a limit point of $(x(c_k), y(c_k))$ then it

is a constrained saddle point for f .

<u>Proof</u>: There is a convergent subsequence of $(x(c_k), y(c_k))$ converges to (x^*, y^*) . For convenience, we suppose $(x(c_k), y(c_k))$ converges and show firstly that (x^*, y^*) satisfies the constraints $g_i(x) = 0$, $1 \leq i \leq k$, $h_j(y) = 0$, $1 \leq j \leq \ell$.

Since $(x(c_k), y(c_k))$ is a saddle point for $L(x, y; c_k)$, then

$$L(x(c_k), y; c_k) \leq L(x, y(c_k), c_k) .$$

In particular, for those $x \in E^n$, $g_i(x) = 0$, $1 \leq i \leq k$ and $y \in E^m$, $h_j(y) = 0$ $1 \leq j \leq \ell$, we have

$$f(x, (c_k), y) + \frac{c_k}{2} \Sigma g_i(x(c_k))^2$$
$$\leq f(x, y(c_k)) - \frac{c_k}{2} \Sigma h_j(y(c_k))^2$$

i.e. $$\frac{c_k}{2} [\Sigma g_i(x(c_k))^2 + \Sigma h_j(y(c_k))^2] \leq f(x, y(c_k)) - f(x(c_k), y) .$$

Since the right hand side of the above inequality is finite as $c_k \to \infty$, we have

$$\Sigma g_i(x(c_k))^2 + \Sigma h_j(y(c_k))^2 \to 0 \quad \text{as} \quad k \to \infty ,$$

i.e. $$\Sigma g_i(x^*)^2 + \Sigma h_j(y^*)^2 = 0 ,$$

or equivalently, $g_i(x^*) = 0$, $1 \leq i \leq k$; $h_j(y^*) = 0$, $1 \leq j \leq \ell$. Furthermore, we have for $x \in E^n$, $g_i(x) = 0$; $y \in E^m$, $h_j(y) = 0$, the following inequality

$$f(x, y^*) - f(x^*, y) \geqslant 0 \quad ,$$

hence

$$f(x^*, y) \leqslant f(x^*, y^*) \leqslant f(x, y^*) \quad ,$$

i.e. (x^*, y^*) is a constrained saddle point for f .

Remark: As we note that in the previous discussion, the proposition is valid only if the multipliers $\bar{\lambda}_1, \ldots, \bar{\lambda}_k \ ; \ \bar{\mu}_1, \ldots, \bar{\mu}_\ell$ is known in advance. This is somewhat difficult and we will investigate it in the next section.

When inequality constraints are present, we are concerned with saddle point problems with inequality constraints. By it we mean to find a saddle point for f subject to $g_i(x) \leqslant 0$, $1 \leqslant i \leqslant k$, $h_j(y) \leqslant 0$, $1 \leqslant j \leqslant \ell$. We note that, if we let $u = (u_1, \ldots, u_k)$; $v = (v_1, \ldots, v_\ell)$ and define:

$$G_i(x, u) = g_i(x) + u_i^2 \qquad 1 \leqslant i \leqslant k \ ,$$

$$H_j(y, v) = h_j(y) + v_j^2 \qquad 1 \leqslant j \leqslant \ell \ .$$

If $g_i(x) \leqslant 0$, then there is $\bar{u}_i$ s.t. $\bar{u}_i^2 = -g_i(x)$ and hence $G_i(x, \bar{u}) = 0$ for any $\bar{u}$ with its i-th component equal to $\bar{u}_i$. Conversely, if $G_i(x, u) = 0$, then $g_i(x) \leqslant 0$. Hence, if $(\bar{x}, \bar{y})$ is a saddle point for f subject to the inequality constraints, then it is also a saddle point of f subject to the equality constraints $G_i(x, u) = 0$ and $H_j(y, v) = 0$, and vise versa. Hence, we can transform our problem with inequality constraints into a problem with equality constraints. So, according to the previous result on saddle point problem with equality constraints, we have

<u>Proposition 8</u>. If a point $(\bar{x}, \bar{u}; \bar{y}, \bar{v}) \in E^n \times E^k \times E^m \times E^\ell$ be such that

(i) $G_i(\bar{x}, \bar{u}) = 0$, $H_j(\bar{y}, \bar{v}) = 0$, $1 \leq i \leq k$, $1 \leq j \leq \ell$,

(ii) $\nabla_{(x, u)} F(\bar{x}, \bar{u}; \bar{y}, \bar{v}) = 0$, $\nabla_{(y, v)} F(\bar{x}, \bar{u}; \bar{y}, \bar{v}) = 0$,

(iii) $h^T \nabla^2_{x,u} F(\bar{x}, \bar{u}; \bar{y}, \bar{v})h > 0$ for all $h \in E^n \times E^k$, $h \neq 0$,

and $\nabla G_i(\bar{x}, \bar{u})^T h = 0$, $1 \leq i \leq k$,

$k^T \nabla^2_{y,v} F(\bar{x}, \bar{u}; \bar{y}, \bar{v})k < 0$ for $k \in E^m \times E^\ell$, $k \neq 0$ and

$\nabla H_j(\bar{y}, \bar{v})^T k = 0$, $1 \leq j \leq \ell$.

Where $F(x, u, y, v) = f(x, y) + \Sigma \lambda_i G_i(x, u) - \Sigma \mu_j H_j(y, v)$. Then there exists a constant $\bar{c} > 0$, s.t. for $c \geq \bar{c}$; $(\bar{x}, \bar{u}; \bar{y}, \bar{v})$ is an unconstrained local saddle point for $\tilde{L}(x, u; y, v, c) = F(x, u, y, v) + \frac{c}{2}[\Sigma G_i(x, u)^2 - \Sigma H_j(y, v)^2]$. Consequently, $(\bar{x}, \bar{y})$ is a local saddle point for f subject to the constraints $g_i(x) \leq 0$, $h_j(y) \leq 0$.

Concerning the multipliers $\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_\ell$, we have the following:

<u>Corollary 9</u>. If $g_i(\bar{x}) = 0$, then $\lambda_i > 0$; if $g_i(\bar{x}) < 0$, then $\lambda_i = 0$. If $h_j(\bar{y}) = 0$, then $\mu_j > 0$, if $h_j(\bar{y}) < 0$, then $\mu_j = 0$.

<u>Proof</u>. From condition (ii), we have

$$\nabla_x f(\bar{x}, \bar{y}) + \Sigma \lambda_i \nabla g_i(\bar{x}) = 0 ,$$

$$2 \lambda_i \bar{u}_i = 0 , \qquad 1 \leq i \leq k ,$$

and $\quad \nabla_y f(\bar{x}, \bar{y}) - \Sigma \mu_j \nabla h_j(\bar{y}) = 0 ,$

$$- 2 \mu_j \bar{v}_j = 0 \qquad 1 \leqslant j \leqslant \ell .$$

Hence, if $\bar{u}_i \neq 0$, i.e. $g_i(\bar{x}) < 0$, then $\lambda_i = 0$ and if $\bar{v}_j \neq 0$, i.e. $h_j(\bar{y}) < 0$, then $\mu_j = 0$.

For the first equation in (iii), we let

$$\nabla_{x,u}{}^2 F(\bar{x}, \bar{u}; \bar{y}, \bar{v}) = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} ,$$

where $A_{11} = \nabla^2_{xx} \, f(\bar{x}, \bar{y}) + \Sigma \, \lambda_i \, \nabla^2 g_i(\bar{x})$,

$$A_{22} = \mathrm{diag}\{2 \lambda_1, \ldots, 2 \lambda_k\}$$

and $h = (h_1, h_2)$, $h_1 \in E^n$, $h_2 \in E^k$, s.t. $h \neq 0$. If $h^T \nabla G_i(\bar{x}, \bar{u}) = 0$, i.e. $h_1{}^T \nabla g_i(\bar{x}) + 2h_2{}^i \bar{u}_i = 0$. We have by (iii)

$$(h_1, h_2)^T \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} (h_1, h_2) = h_1{}^T A_{11} h + h_2{}^T A_{22} h_2 > 0 .$$

If $g_i(\bar{x}) = 0$, then $\bar{u}_i = 0$; hence the vector $h = (h_1, h_2)$, with $h_1 = 0$, $h_2 = (0, \ldots, 1, 0, \ldots, 0)$ with its i-th component, equals to 1 and zero otherwise, satisfies $h_1{}^T \nabla g_i(\bar{x}) + 2h_2{}^i \bar{u}_i = 0$ for all $1 \leqslant i \leqslant k$. Hence $2 \lambda_i > 0$, i.e. $\lambda_i > 0$. Similarly, if $h_j(\bar{y}) = 0$ then $\mu_j > 0$.

SECTION 3.

We have mentioned in the previous section that, if to our saddle point problem, we know in advance the multipliers $\bar{\lambda}_1, \ldots, \bar{\lambda}_k$; $\bar{\mu}_1, \ldots, \bar{\mu}_\ell$, then an algorithm for searching the constrained saddle points of f can be derived. In practice, there is no a priori information about what the multipliers would be. The following lemma suggests that we can approach these multipliers from an arbitrary multipliers $\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_\ell$ without knowing $\bar{\lambda}_1, \ldots, \bar{\lambda}_k, \bar{\mu}_1, \ldots, \bar{\mu}_\ell$ exactly.

<u>Lemma 10</u>. Let $\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_\ell$ be arbitrary multipliers. Define

$$L(x, y, \lambda, \mu, c) = f(x, y) + \Sigma\, \lambda_i g_i(x) - \Sigma\, \mu_j h_j(y) + \frac{c}{2}[\Sigma\, g_i^{\,2}(x) - \Sigma\, h_j^{\,2}(y)] ,$$

and we suppose for each $c > 0$, $L(x, y, \lambda, \mu, c)$ has a saddle point $(x(c), y(c))$, and $(x(c), y(c)) \to (\bar{x}, \bar{y})$ as $c \to \infty$.

Let $\{ \nabla g_1(\bar{x}), \ldots, \nabla g_k(\bar{x})\}, \{\nabla h_1(\bar{y}), \ldots, \nabla h_\ell(\bar{y})\}$ be two linearly independent sets. Then, as $c \to \infty$,

$$\lambda_i + cg_i(x(c)) \to \bar{\lambda}_i \qquad 1 \leq i \leq k$$

$$\mu_j + ch_j(y(c)) \to \bar{\mu}_j \qquad 1 \leq j \leq \ell \quad .$$

<u>Proof</u>. We note that, by proposition 7, $(\bar{x}, \bar{y})$ is a saddle point for f subject to $g_i(x) = 0$ and $h_j(y) = 0$. Hence

$$\nabla_x f(\bar{x}, \bar{y}) + \Sigma\, \bar{\lambda}_i \nabla g_i(\bar{x}) = 0 ,$$
$$\nabla_y f(\bar{x}, \bar{y}) - \Sigma\, \bar{\mu}_j \nabla h_j(\bar{y}) = 0 .$$

Since $(x(c), y(c))$ is a saddle point for $L(x, y, \lambda, \mu, c)$, we have $\nabla_y f(x(c), y(c)) - \Sigma(\mu_j + ch_j(y(c)))\nabla h_j(y(c)) = 0$. We investigate the first equality only, and the discussion for the second equality is the same. For simplicity, the first equation can be rewritten in the matrix form:

$$\nabla g(x(c))(\lambda_1 + cg_1(x(c)), \ldots, \lambda_k + cg_k(x(c))) = -\nabla_x f(x(c), y(c)) .$$

Here $\nabla g(x(c))$ is the matrix having $\nabla g_i(x(c))$ as its i-th column. From the fact that $\{\nabla g_1(\bar{x}), \ldots, \nabla g_k(\bar{x})\}$ is linearly independent and that $x(c) \to \bar{x}$ as $c \to \infty$, we have for sufficiently large c, $\nabla g(x(c))^T \nabla g(x(c))$ is non-singular. Hence for sufficiently large c,

$$(\lambda_1 + cg_1(x(c)), \ldots, \lambda_k + cg_k(x(c)) = -[\nabla g(x(c))^T \nabla g(x(c))]^{-1} \nabla g(x(c))^T \nabla_x f(x(c), y(c)).$$

Since, $(\bar{\lambda}_1, \ldots, \bar{\lambda}_k) = -[\nabla g(\bar{x})^T \nabla g(\bar{x})]^{-1} \nabla g(\bar{x})^T \nabla_x f(\bar{x}, \bar{y})$, we see that $\lambda_i + cg_i(x(c)) \to \bar{\lambda}_i$ as $c \to \infty$. Similarly, $\mu_j + ch_j(y(c)) \to \bar{\mu}_j$ as $c \to \infty$.

Due to this lemma, we are led to the following consideration: starting from arbitrary multipliers λ^o, μ^o; in stead of leaving them fixed, we update them in the k-th step by the following rule:

$$\lambda_i^k = \lambda_i^{k-1} + c_{k-1} g_i(x_{k-1}) \quad 1 \leqslant i \leqslant k ,$$
$$\mu_j^k = \mu_j^{k-1} + c_{k-1} h_j(y_{k-1}) \quad 1 \leqslant j \leqslant \ell ,$$

where (x_{k-1}, y_{k-1}) is a saddle point for $L(x, y, \lambda^{k-1}, \mu^{k-1}, c_{k-1})$. Is it true that, without leting c_k diverges to infinite, a constrained saddle point of f could be obtained? i.e. if $c_k \to c^* < +\infty$ as $k \to \infty$, $(x_k, y_k) \to (x^*, y^*)$, then (x^*, y^*) is a constrained saddle point for f. The answer to this question is positive. In fact, we shall show that: There exists a constant $\bar{c} > 0$, s.t. for every $c \geq \bar{c}$, and multipliers λ, μ, $L(x, y, \lambda, \mu, c)$ has a unique saddle point $(x(c), y(c))$ within some open ball centered at $(\bar{x}, \bar{y})$. Furthermore, there exists a constant $M > 0$ s.t.

$$\|(x(c), y(c)) - (\bar{x}, \bar{y})\| \leq \frac{M\|\bar{\eta} - \eta\|}{c}$$

and

$$\|(\lambda(c), \mu(c)) - (\bar{\lambda}, \bar{\mu})\| \leq \frac{M\|\bar{\eta} - \eta\|}{c},$$

where
$$\lambda(c)^i = \lambda^i + cg_i(x(c)), \tag{*}$$

$$\mu(c)^j = \mu^j + ch_j(y(c)),$$

$$\eta = (\lambda, \mu), \quad \bar{\eta} = (\bar{\lambda}, \bar{\mu}).$$

Once the validity of this theorem is proved, some important conclusion can be derived. If $\lambda = 0$, $\mu = 0$, this is equivalent to the method of penalty functions. If the vector (λ, μ) is not held fixed but rather is updated by means of the iteration

$$\begin{aligned} \lambda_{k+1} &= \lambda_k + c_k\, g(x(c_k)), \\ \mu_{k+1} &= \mu_k + c_k\, h(y(c_k)), \end{aligned} \tag{5}$$

this is called the multiplier method provided the λ_{k+1}, μ_{k+1} remains finite. In order it is so, we specify a bounded subset S of $E^n \times E^m$, with

$(\lambda^o, \mu^o) \in S$, we require the updating takes place provided the resulting vector $(\lambda_{k+1}, \mu_{k+1})$ belongs to S . Otherwise, $\lambda_{k+1} = \lambda_k, \mu_{k+1} = \mu_k$; i.e. λ_k, μ_k left unchanged. In this case, we have that, if $c_k \to \infty$ as $k \to \infty$, then

$$\lim_{k\to\infty} \frac{\|(\lambda_{k+1}, \mu_{k+1}) - (\bar{\lambda}, \bar{\mu})\|}{\|(\lambda_k, \mu_k) - (\bar{\lambda}, \bar{\mu})\|} = 0 ,$$

i.e. the sequence $\{(\lambda_k, \mu_k)\}$ converges to $(\bar{\lambda}, \bar{\mu})$ superlinearly. If $c_k \to c < \infty$, where c is sufficiently large (large enough to ensure that $c > M$, $c > \bar{c}$, and $\lambda_k + c\, g(x(c_k)), \mu_k + ch(y(c_k))$ belongs to an open sphere centered at $(\bar{\lambda}, \bar{\mu})$ and contained in S) then

$$\overline{\lim_{k\to\infty}} \frac{\|(\lambda_{k+1}, \mu_{k+1}) - (\bar{\lambda}, \bar{\mu})\|}{\|(\lambda_k, \mu_k) - (\bar{\lambda}, \bar{\mu})\|} \leqslant \frac{M}{c} ,$$

i.e. $\{(\lambda_k, \mu_k)\}$ converges to $(\bar{\lambda}, \bar{\mu})$ at least linearly with a convergence ratio inversely proportional to c . The method of updating (λ, μ) can be viewed as a generalized multiplier method for optimization problems.

In conclusion the method of multiplier defined by (5) converges from an arbitrary starting point within the bounded S provided that c_k is sufficiently large after some index $\bar{k}$; $(\bar{\lambda}, \bar{\mu})$ is an interior point of S and the unconstrained saddle point $(x(c_k), y(c_k))$ exists for all $k \geqslant \bar{k}$. In addition, the multiplier method offers distinct advantages over the quadratic penalty method in that it avoids the necessity of increasing c_k to infinity and furthermore, the estimate of its convergent rate is much more favourable. For we see that

$$\|(\lambda_k, \mu_k) - (\bar{\lambda}, \bar{\mu})\| \leq \|(\lambda^o, \mu^o) - (\bar{\lambda}, \bar{\mu})\| \cdot \prod_{i=1}^{k} \frac{M}{c_i} .$$

and therefore,

$$\|(x(c_k), y(c_k)) - (\bar{x}, \bar{y})\| \leq \|(\lambda^o, \mu^o) - (\bar{\lambda}, \bar{\mu})\| \prod_{i=1}^{k} \frac{M}{c_i} .$$

But in the penalty function method, we have

$$\|(x(c_k), y(c_k)) - (\bar{x}, \bar{y})\| \leq \frac{M}{c_i} \|(\bar{\lambda}, \bar{\mu})\| .$$

If we set $\lambda^o = 0$, $\mu^o = 0$, in the multiplier method, we have

$$\|(x(c_k), y(c_k)) - (\bar{x}, \bar{y})\| \leq \|(\bar{\lambda}, \bar{\mu})\| \prod_{i=1}^{k} \frac{M}{c_i} .$$

The ratio of the two bounds is $\prod_{i=1}^{k-1} \frac{M}{c_i}$ and it tends to zero as $k \to \infty$.

Before proving the theorem, we assume the following :

(i) $\nabla^2_{xy} f(\bar{x}, \bar{y}) = 0$, $\nabla^2_{yx} f(\bar{x}, \bar{y}) = 0$,

(ii) There exists multipliers $\bar{\lambda}_1, \ldots, \bar{\lambda}_k ; \bar{\mu}_1, \ldots, \bar{\mu}_\ell$ s.t.

$$\nabla_x f(\bar{x}, \bar{y}) + \Sigma \bar{\lambda}_i \nabla g_i(\bar{x}) = 0 ,$$

$$\nabla_y f(\bar{x}, \bar{y}) - \Sigma \bar{\mu}_j \nabla h_j(\bar{y}) = 0 ;$$

(iii) Let $L_o(x, y, \bar{\lambda}, \bar{\mu}) = f(x, y) + \Sigma \bar{\lambda}_i g_i(x) - \Sigma \bar{\mu}_j h_j(y)$ and the matrix $\nabla^2_{xx} L_o(\bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu})$ is positive definite on the tangent plane associated with $\nabla g_i(\bar{x})$, $1 \leq i \leq k$, i.e. for $h \neq 0$, $h^T \nabla g_i(\bar{x}) = 0$, $1 \leq i \leq k$, implies $h^T \nabla^2_{xx} L_o(\bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu}) h > 0$. Likewise, if $k \neq 0$, $k^T \nabla h_j(\bar{y}) = 0$ for $1 \leq j \leq \ell$, we have $k^T \nabla^2_{yy} L_o(\bar{x}, \bar{y}; \bar{\lambda}, \bar{\mu}) k < 0$;

(iv) The functions $\nabla^2 f(x, y)$, $\nabla^2 g_i(x)$, $\nabla^2 h_j(y)$ satisfy the Lipschitz condition within some open ball centered at $(\bar{x}, \bar{y})$. i.e. for some $K > 0$, for every (x_1, y_2), (x_2, y_2) in $B((\bar{x}, \bar{y}); \varepsilon)$, we have

$$\|\nabla^2 f(x_1, y_1) - \nabla^2 f(x_2, y_2)\| \leqslant K\|(x_1, y_1) - (x_2, y_2)\| ,$$

$$\|\nabla^2 g_i(x_1) - \nabla^2 g_i(x_2)\| \leqslant K\|x_1 - x_2\| ,$$

$$\|\nabla^2 h_j(y_1) - \nabla^2 h_j(y_2)\| \leqslant K\|y_1 - y_2\| .$$

Under the hypothesis (i) to (iv) cited above, we have

Theorem. There exists a constant $\bar{c} > 0$, s.t. for every $c \geqslant \bar{c}$, and multipliers λ, μ ; $L(x, y, \lambda, \mu, c)$ has a unique saddle point $(x(c), y(c))$ within some open ball centered at $(\bar{x}, \bar{y})$. Furthermore, there is a constant $M > 0$, s.t.

$$\|(x(c), y(c)) - (\bar{x}, \bar{y})\| \leqslant \frac{M\|\bar{\eta} - \eta\|}{c} ,$$

$$\|(\lambda(c), \mu(c)) - (\bar{\lambda}, \bar{\mu})\| \leqslant \frac{M\|\bar{\eta} - \eta\|}{c} ,$$

where $\lambda(c), \mu(c)$, η , $\bar{\eta}$ is defined by (*) .

Proof. We first prove our theorem under the assumption

(c): $\nabla_{xx}^2 L_o(\bar{x}, \bar{y}; \bar{\lambda}, \bar{\mu})$ is positive definite,

$\nabla_{yy}^2 L_o(\bar{x}, \bar{y}; \bar{\lambda}, \bar{\mu})$ is negative definite.

For $(x, y) \in B((\bar{x}, \bar{y}); \varepsilon)$ fixed $(\lambda, \mu) \in S$, $c > 0$, we let

$p = (x - \bar{x}, y - \bar{y})^T$, $q = (q_1, q_2)^T = (\lambda + cg(x) - \bar{\lambda}, \mu + ch(y) - \bar{\mu})^T$,

i.e. $q_1{}^i = \lambda_i + cq_i(x) - \bar{\lambda}_i$; $q_2{}^j = \mu_j + ch_j(y) - \bar{\mu}_j$.

Now
$$\nabla f(x, y) = \nabla f(\bar{x}, \bar{y}) + \nabla^2 f(\bar{x}, \bar{y})p + r_1(p) ,$$
$$\nabla g_i(x) = \nabla g_i(\bar{x}) + \nabla^2 g_i(\bar{x})(x - \bar{x}) + r_2{}^i(x - \bar{x}) ,$$
$$\nabla h_j(y) = \nabla h_j(\bar{y}) + \nabla^2 h_j(\bar{y})(y - \bar{y}) + r_2{}^j(y - \bar{y}) ,$$

where $r_1(0) = r_2{}^i(0) = r_2{}^j(0) = 0$, for all i, j .

Since $\nabla r_1(p) = \nabla^2 f(x, y) - \nabla^2 f(\bar{x}, \bar{y})$,

by assumption (iii) $\|\nabla r_1(p)\| \leq K \|p\|$.

Similarly, we have $\|\nabla r_2{}^i(x - \bar{x})\| \leq K \|x - \bar{x}\|$

and $\|\nabla r_2{}^j(y - \bar{y})\| \leq K \|y - \bar{y}\|$.

Note that

$$\nabla L(x, y, \lambda, \mu, c) = \nabla f(x, y) + \begin{bmatrix} \Sigma(\lambda_i + cg_i(x))\nabla g_i(x) \\ -\Sigma(\mu_j + ch_j(y))\nabla h_j(y) \end{bmatrix} .$$

By substitution, we have

$$\nabla L(x, y, \lambda, \mu, c) = \nabla f(\bar{x}, \bar{y}) + \nabla^2 f(\bar{x}, \bar{y})p + r_1(p)$$
$$+ \begin{bmatrix} \Sigma(q^i + \bar{\lambda}_i)(\nabla g_i(\bar{x}) + \nabla^2 g_i(\bar{x})(x - \bar{x}) + r_2{}^i(x - \bar{x})) \\ -\Sigma(q^i + \bar{\mu}_j)(\nabla h_j(\bar{y}) + \nabla^2 h_j(\bar{y})(y - \bar{y}) + r_2{}^j(y - \bar{y})) \end{bmatrix} .$$

From hypothesis (ii), we have

$$\nabla L(x, y, \lambda, \mu, c) = \nabla^2 f(\bar{x}, \bar{y})p + \begin{bmatrix} \Sigma \bar{\lambda}_i \nabla^2 g_i(\bar{x}) & 0 \\ 0 & -\Sigma \bar{\mu}_j \nabla^2 h_j(\bar{y}) \end{bmatrix} p$$

Therefore $\|\nabla r_4^{\,i}(x - \bar{x})\| \leq \|\nabla^2 g_i(\bar{x})(x - \bar{x}) + r_2^{\,i}(x - \bar{x})\|$

$$\leq (\|\nabla^2 g_i(\bar{x})\| + K)\,\|x - \bar{x}\| \ .$$

Similarly,

$$\|\nabla r_4^{\,j}(y - \bar{y})\| \leq (\|\nabla^2 h_j(\bar{y})\| + K)\|y - \bar{y}\| \ ,$$

i.e. $\exists$ constant K' , s.t. $\|\nabla r_4(p)\| \leq K'\|p\|$.

Combining now, we have that in order for a point (x, y) in $B((\bar{x}, \bar{y}); \varepsilon)$ to satisfy $\nabla L(x, y, \lambda, \mu, c) = 0$, it is necessary and sufficient that the corresponding point $S = \begin{bmatrix} p \\ q \end{bmatrix}$ solves the equation

$$A\,s = t + r(s) \tag{6}$$

where

$$A = \begin{bmatrix} \nabla^2 L_0(\bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu}) & \begin{bmatrix} \nabla g(\bar{x}) & 0 \\ 0 & -\nabla h(\bar{y}) \end{bmatrix} \\ \begin{bmatrix} \nabla g(\bar{x})^T & 0 \\ 0 & \nabla h(\bar{y})^T \end{bmatrix} & -\dfrac{I}{c} \end{bmatrix}$$

$$t = \frac{\bar{\eta} - \eta}{c} \ , \qquad r(s) = \begin{bmatrix} -r_3(p, q) \\ r_4\ (p) \end{bmatrix} ,$$

where I is an $(k + \ell) \times (k + \ell)$ identity matrix. The vector $r(s)$ has the property that $\|\nabla r(s)\| \leq \alpha\|s\|$ and $r(0) = 0$ where α is a constant depend on ε .

For the equation (6), we have the following results:

(a) The matrix A has an inverse for each $c > 0$ and there exists a constant M s.t. $\|A^{-1}\| \leq M$ for large c.

(b) There is a $\bar{c} > 0$, s.t. for all $c \geq \bar{c}$, $\exists$ a unique solution $s^* \in B(0; \varepsilon)$ to $As = t + r(s)$ and that $\|s^*\| \leq M\|t\|$ (Note that t depend on c).

Following from these results, we have, $\exists\ \bar{c} > 0$, s.t. for all $c \geq \bar{c}$, there is a unique solution $(x(c), y(c))$ within some open ball centered at $(\bar{x}, \bar{y})$, s.t. $\nabla L(x(c), y(c), \lambda, \mu, c) = 0$. Furthermore,

$$\|(x(c), y(c)) - (\bar{x}, \bar{y})\| \leq \frac{M\|\bar{\eta} - \eta\|}{c},$$

$$\|(\lambda(c), \mu(c)) - (\bar{\lambda}, \bar{\mu})\| \leq \frac{M\|\bar{\eta} - \eta\|}{c}.$$

In fact, $(x(c), y(c))$ is a saddle point for $L(x, y, \lambda, \mu, c)$. To this end, it is sufficient to show that the sufficient condition is also satisfied. Now

$$\begin{aligned}\nabla^2_{xx} L(x(c), y(c), \lambda, \mu, c) = {} & \nabla^2_{xx} f(x(c), y(c)) \\ & + \Sigma\,(\lambda^i + c\ g_i(x(c)))\ \nabla g_i{}^2(x(c)) \\ & + c\ \nabla g(x(c))\ \nabla g(x(c))^T .\end{aligned}$$

As $c \to \infty$, $(x(c), y(c)) \to (\bar{x}, \bar{y})$ as well as $\lambda_i + c\ g_i(x(c)) \to \bar{\lambda}_i$ and since $\nabla^2_{xx} f(\bar{x}, \bar{y}) + \Sigma\ \bar{\lambda}_i \nabla^2 g_i(\bar{x})$ is positive definite so that for sufficiently large c,

$$\nabla^2_{xx} f(\bar{x}, \bar{y}) + \Sigma\,(\lambda^i + cg_i(x(c)))\ \nabla^2 g_i(x(c))$$

is positive definite. Note also $\nabla g(x(c))\nabla g(x(c))^T$ is positive semi-definite, then $\nabla^2_{xx} L(x(c), \lambda, \mu, c)$ is positive definite. Similarly, $\nabla^2_{yy} L(x(c), y(c), \lambda, \mu, c)$ is negative definite for sufficiently large c. That is, $(x(c), y(c))$ is a saddle point for $L(x, y, \lambda, \mu, c)$ for large c.

Remember that our theorem is establish on the hypothesis (c). In the general case, i.e. if hypothesis (iii) holds, we see that for $\sigma > 0$, the problem:

Find a saddle point for $f(x, y) + \frac{\sigma}{2}[\Sigma\, g_i(x)^2 - \Sigma h_j(y)^2]$ subject to the constraints $g_i(x) = 0$ and $h_j(y) = 0$

is equivalent to our original problem. Hence, we may substitute our original function $f(x, y)$ by

$$\bar{f}(x, y, \sigma) = f(x, y) + \frac{\sigma}{2}[\Sigma\, g_i(x)^2 - \Sigma\, h_j(y)^2] .$$

We note that, if we let

$$\bar{L}(x, y, \lambda, \mu, \sigma) = \bar{f}(x, y, \sigma) + \Sigma\, \lambda_i g_i(x) - \Sigma\, \mu_j h_j(y) ,$$

then
$$\nabla^2_{xx} \bar{L}(\bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu}, \sigma) = \nabla^2_{xx} \bar{f}(\bar{x}, \bar{y}, \sigma) + \Sigma\, \bar{\lambda}_i\, \nabla^2 g_i(\bar{x})$$

and
$$\nabla^2_{xx} \bar{f}(\bar{x}, \bar{y}, \sigma) = \nabla^2_{xx} f(x, y) + \sigma\, \nabla g_i(\bar{x})\, \nabla g_i(\bar{x})^T .$$

Since, by hythesis (iii), and lemma 5, we have, $\exists\, \bar{\sigma} > 0$ s.t. $\forall\, \sigma \geq \bar{\sigma}$, $\nabla_{xx}\bar{L}(\bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu}, \sigma)$ is positive definite. Similarly, $\nabla_{yy}\bar{L}(\bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu}, \sigma)$ is negative definite for large σ. Whence if we take $\bar{f}(x, y, \bar{\sigma})$ instead of $f(x, y)$ in our problem, then hypothesis (c) holds. Accordingly, we have

$$L(x, y, \lambda, \mu, c) = \bar{L}(x, y, \lambda, \mu, \bar{\sigma}) + \frac{c - \bar{\sigma}}{2}[\Sigma\, g_i(x)^2 - \Sigma h_j(y)^2]$$

so that $\exists$ a $\bar{c} > 0$, s.t. $\forall\, c \geq \bar{c} - \bar{\sigma}$, $\exists$ unique $(x(c), y(c))$, a

saddle point of $L(x, y, \lambda, \mu, c)$, within some open ball centered at $(\bar{x}, \bar{y})$.
Furthermore, $\exists\ M > 0$ s.t.

$$\|(x(c), y(c)) - (\bar{x}, \bar{y})\| \leq \frac{M\|\bar{\eta} - \eta\|}{c - \bar{\sigma}}$$

and $\|(\lambda(c), \mu(c)) - (\bar{\lambda}, \bar{\mu}) - (\bar{\sigma} g(x(c))\ \bar{\sigma} h(y(c))\| \leq \dfrac{M\|\bar{\eta} - \eta\|}{c - \bar{\sigma}}$.

For sufficiently large c , $\|(g(x(c)), h(y(c))\| \leq \dfrac{BM\|\bar{\eta} - \eta\|}{c - \bar{\sigma}}$, B a constant,
hence

$$\|(\lambda(c), \mu(c)) - (\bar{\lambda}, \bar{\mu})\| \leq \frac{(\bar{\sigma}B + 1)M\|\bar{\eta} - \eta\|}{c - \bar{\sigma}} .$$

The proof is completed.

In our proof, we have made use of the fact that A^{-1} exists for all $c > 0$ and is uniformly bounded for large c . Furthermore, $\exists\ \bar{c} > 0$, s.t. $\forall\ c > \bar{c}$, there is a unique solution s^* to $As = t + r(s)$ within some open ball centered at 0 , and that $\|s^*\| \leq M\|t\|$. We are now ready to prove these results. First, we show that A^{-1} do exists for all $c > 0$.

We let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where $A_{11} = \nabla^2 L_o(\bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu})$

$$= \begin{bmatrix} \nabla_{xx}^2 f(\bar{x}, \bar{y}) + \Sigma\, \bar{\lambda}_i \nabla^2 g_i(\bar{x}) & 0 \\ 0 & \nabla_{yy}^2 f(\bar{x}, \bar{y}) - \Sigma\, \bar{\mu}_j \nabla^2 h_j(\bar{y}) \end{bmatrix} ,$$

$$A_{12} = \begin{bmatrix} \nabla g(\bar{x}) & 0 \\ 0 & -\nabla h(\bar{y}) \end{bmatrix}, \qquad A_{21} = \begin{bmatrix} \nabla g(\bar{x})^T & 0 \\ 0 & \nabla h(\bar{y})^T \end{bmatrix}$$

$$A_{22} = -\frac{1}{c} I .$$

Then $\det A = \det(A_{11} - A_{12}A_{22}^{-1}A_{21}) \times \det A_{22}$,

Since $A_{22}^{-1} = [-\frac{1}{c} I]^{-1} = -c I$ and $\det A_{22} = -c \neq 0$,

$$A_{11} - A_{12}A_{22}^{-1}A_{21} = A_{11} + c A_{12} A_{21}$$

$$= \begin{bmatrix} \nabla_{xx}^2 f(\bar{x}, \bar{y}) + \Sigma \bar{\lambda}_i \nabla^2 g_i(\bar{x}) + c \nabla g(\bar{x}) \nabla g(\bar{x})^T & 0 \\ 0 & \nabla_{yy}^2 f(\bar{x}, \bar{y}) - \Sigma \bar{\mu}_j \nabla^2 h_j(\bar{y}) - c \nabla h(\bar{y}) \nabla h(\bar{y})^T \end{bmatrix} .$$

Note that $\nabla_{xx}^2 f(\bar{x}, \bar{y}) + \Sigma \bar{\lambda}_i \nabla^2 g_i(\bar{x})$ is assumed to be positive definite, and $\nabla g(\bar{x}) \nabla g(\bar{x})^T$ is positive semi-definite, then its determinent is nonzero. Similarly, the determinent of $\nabla_{yy}^2 f(\bar{x}, \bar{y}) - \Sigma \bar{\mu}_j \nabla^2 h(\bar{y}) - c \nabla h(\bar{y}) \nabla h(\bar{y})^T$ is non-zero. It follows that $\det(A_{11} - A_{12} A_{22}^{-1} A_{21}) \neq 0$. Hence $\det A \neq 0$, i.e. A^{-1} exists for all $c > 0$.

To show that A^{-1} is uniformly bounded, we note that

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1} A_{12} \tilde{A}_{22}^{-1} \\ 0 & \tilde{A}_{22}^{-1} \end{bmatrix} \begin{bmatrix} I_{n+m} & 0 \\ -A_{21}A_{11}^{-1} & I_{k+\ell} \end{bmatrix}$$

where $\tilde{A}_{22} = A_{22} - A_{21} A_{11}^{-1} A_{12}$.

We set $A_{11} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$,

$$B_{11} = \nabla^2_{xx} f(\bar{x}, \bar{y}) + \Sigma \bar{\lambda}_i \nabla^2 g_i(\bar{x}) , \quad B_{12} = B_{21} = 0 ,$$

$$B_{22} = \nabla^2_{yy} f(\bar{x}, \bar{y}) - \Sigma \bar{\mu}_j \nabla^2 h_j(\bar{y}) .$$

Then $A_{11}^{-1} = \begin{bmatrix} B_{11}^{-1} & 0 \\ 0 & B_{22}^{-1} \end{bmatrix}$,

$$A_{21} A_{11}^{-1} A_{12} = \begin{bmatrix} \nabla g(\bar{x})^T & 0 \\ 0 & \nabla h(\bar{y})^T \end{bmatrix} \begin{bmatrix} B_{11}^{-1} & 0 \\ 0 & B_{22}^{-1} \end{bmatrix} \begin{bmatrix} \nabla g(\bar{x}) & 0 \\ 0 & -\nabla h(\bar{y}) \end{bmatrix}$$

$$= \begin{bmatrix} \nabla g(\bar{x})^T B_{11}^{-1} \nabla g(\bar{x}) & 0 \\ 0 & -\nabla h(\bar{y})^T B_{22}^{-1} \nabla h(\bar{y}) \end{bmatrix} .$$

Therefore,

$$\tilde{A}_{22} = A_{22} - A_{21} A_{11}^{-1} A_{12}$$

$$= \begin{bmatrix} -\frac{1}{c} I_k - \nabla g(\bar{x})^T B_{11}^{-1} \nabla g(\bar{x}) & 0 \\ 0 & -\frac{1}{c} I_\ell + \nabla h(\bar{y}) B_{22}^{-1} \nabla h(\bar{y}) \end{bmatrix} .$$

Since B_{11} is positive definite, so is B_{11}^{-1}. It follows that $-(\frac{1}{c} I_k + \nabla g(\bar{x})^T B_{11}^{-1} \nabla g(\bar{x}))$ is negative definite; similarly, $-(\frac{1}{c} I_\ell - \nabla h(\bar{y}) B_{22}^{-1} \nabla h(\bar{y}))$ is negative definite, hence the inverse of $\tilde{A}_{22}$ exists, and equals to

$$\begin{bmatrix} (-\frac{1}{c} I_k - \nabla g(\bar{x})^T B_{11}^{-1} \nabla g(\bar{x}))^{-1} & 0 \\ 0 & (-\frac{1}{c} I_\ell + \nabla h(\bar{y}) B_{22}^{-1} \nabla h(\bar{y}))^{-1} \end{bmatrix} .$$

In order to compute the norm of $\tilde{A}_{22}^{-1}$, we transform the matrices $\nabla g(\bar{x})^T B_{11}^{-1} \nabla g(\bar{x})$ and $\nabla h(\bar{y}) B_{22}^{-1} \nabla h(\bar{y})$ into their Jordan canonical forms. Thus there exist non-singular matrices P and Q, s.t.

$$P^{-1}\nabla g(\bar{x})^T B_{11}^{-1}\nabla g(\bar{x})P = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{bmatrix},$$

$$Q^{-1}\nabla h(\bar{y})^T B_{22}^{-1}\nabla h(\bar{y})Q = \begin{bmatrix} J'_1 & & & \\ & J'_2 & & \\ & & \ddots & \\ & & & J'_s \end{bmatrix},$$

where the submatrices J_i and J'_j are of the form

$$\begin{bmatrix} t & & 0 \\ 1 & \ddots & \\ & \ddots & \ddots & \\ 0 & & 1 & t \end{bmatrix}.$$

Hence

$$-\frac{1}{c} I_k - \nabla g(\bar{x})^T B_{11}^{-1}\nabla g(\bar{x}) = P(-\frac{1}{c} I_k - P^{-1}\nabla g(\bar{x})^T B_{11}^{-1}\nabla g(\bar{x})P)P^{-1}$$

$$= P \begin{bmatrix} -\frac{1}{c} I_1 - J_1 & & & \\ & -\frac{1}{c} I_2 - J_2 & & \\ & & \ddots & \\ & & & -\frac{1}{c} I_r - J_r \end{bmatrix} P^{-1}.$$

Then $(-\frac{1}{c} I_k - \nabla g(\bar{x})^T B_{11}^{-1}\nabla g(\bar{x}))^{-1}$

$$= P^{-1} \begin{bmatrix} (-\frac{1}{c} I_1 - J_1)^{-1} & & & \\ & (-\frac{1}{c} I_2 - J_2)^{-1} & & \\ & & \ddots & \\ & & & (-\frac{1}{c} I_r - J_r)^{-1} \end{bmatrix} P,$$

where

$$\left(-\frac{1}{c} I_i - J_i\right)^{-1} = \begin{bmatrix} -\frac{1}{c} - t_i & & & 0 \\ 1 & -\frac{1}{c} - t_i & & \\ & 1 & \ddots & \\ 0 & & 1 & -\frac{1}{c} - t_i \end{bmatrix}^{-1} .$$

If we let $a_i = \dfrac{-1}{\frac{1}{c} + t_i}$, then

$$\left(-\frac{1}{c} I_i - J_i\right)^{-1} = \begin{bmatrix} a_i & & & & 0 \\ -a_i^2 & a_i & & & \\ a_i^3 & -a_i^2 & a_i & & \\ \vdots & & & \ddots & \\ (-1)^{1+i} a_i^{\,i} & \cdots & \cdots & \cdots & a_i \end{bmatrix}_{i \times i} .$$

Since for any $c > 0$, $a_i^2 \leqslant m_i$, a constant, therefore $\|(-\frac{1}{c} I_i - J_i)^{-1}\|^2 =$ sum of square of all entries $\leqslant M_i$. In order that a_i is defined for all i , and note that t_i never vanish, we may take c to be so large that $-\frac{1}{c} \neq t_i$, for each i . Consequently, for a large i ,

$$\begin{aligned} \|(-\tfrac{1}{c} I_k - \nabla g(\bar{x})^T B_{11}^{-1} \nabla g(\bar{x})^{-1}\|^2 &\leqslant \|P\|^2 \, \|P^{-1}\|^2 \, (\Sigma \|-\tfrac{1}{c} I_i - J_i)^{-1}\|^2) \\ &= \|P\|^2 \, \|P^{-1}\|^2 (\Sigma M_i) \\ &= M_1 \ . \end{aligned}$$

Similarly, if c is large,

$$\|(-\frac{1}{c} I_\ell + \nabla h(\bar{y})^T B_{22}^{-1} \nabla h(\bar{y})^{-1}\|^2 \leqslant M_2 \ , \text{ a constant.}$$

Finally $\|\tilde{A}_{22}^{-1}\|^2 \leq M_1 + M_2$ for large c . Returning to A^{-1} , we have

$$\|A^{-1}\|^2 \leq ((\|A_{11}^{-1}\|^2 + \|A_{11}^{-1}\|^2\|A_{12}\|^2\|\tilde{A}_{22}^{-1}\|^2) + \|\tilde{A}_{22}^{-1}\|^2) \times$$
$$((n + m + k + \ell) + \|A_{21}\|^2\|A_{11}^{-1}\|^2) .$$

It follows that for large c , $\|A^{-1}\| \leq M$, M constant.

For each $c > 0$, we consider the mapping

$$F(s, t) = As - t - r(s) ,$$

which has the property that $F(0, 0) = Ao - 0 - r(0) = 0$ and $\nabla_s F(0, 0) = A - \nabla r(0) = A$, since $\|\nabla r(s)\| \leq \alpha\|s\|$. And A is non-singular. Hence by the implicit function Theorem, there are open ball $B(0; \delta)$ and $B(0; \varepsilon)$, s.t. for each $t \in B(0; \delta)$, there corresponds a unique $s^*(t) \in B(0; \varepsilon)$ s.t. when $t = 0$, $s^*(0) = 0$. Furthermore, $As^*(t) = t + r(s^*(t))$.

Differentiating with respect to t , we have

$$A \nabla_t s^*(t) = I + \nabla r(s^*(t)) \cdot \nabla_t s^*(t) .$$

Which implies at $t = 0$, $A \nabla_t s^*(0) = I$, i.e. $\nabla_t s^*(0) = A^{-1}$. Hence, $s^*(t) - s^*(0) = \nabla_t s^*(0) \cdot t$ for t near 0 . In other words, $s^*(t) = A^{-1}t$ for t near 0 . Hence $\|s^*(t)\| \leq \|A^{-1}\|\,\|t\| \leq M\|t\|$. But $t = \frac{\bar{\eta} - \eta}{c}$, we therefore conclude that if c is sufficiently large, then $t \in B(0; \delta)$ and a unique $s^*(t) \in B(0; \varepsilon)$, s.t. $s^*(t)$ is a solution to $As = t + r(s)$ and furthermore $\|s^*(t)\| \leq M\|t\|$. This completes the proof of our second assertion.

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